

# Finite-size scaling analysis of binary stochastic processes and universality classes of information cascade phase transition

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We propose a finite-size scaling analysis of binary stochastic processes  $X(t) \in \{0, 1\}$  based on the second moment correlation length  $\xi$  for the autocorrelation function  $C(t)$ . The purpose is to clarify the critical properties and provide a new data analysis method for information cascades. As a simple model to represent the different behaviors of subjects in information cascade experiments, we assume that  $X(t)$  is a mixture of an independent random variable that takes 1 with probability  $q$  and a random variable that depends on the ratio  $z$  of the variables taking 1 among recent  $r$  variables. We consider two types of the probability  $f(z)$  that the latter takes 1: (i) analog [ $f(z) = z$ ] and (ii) digital [ $f(z) = \theta(z - 1/2)$ ]. We study the universal functions of scaling for  $\xi$  and the integrated correlation time  $\tau$ . For finite  $r$ ,  $C(t)$  decays exponentially as a function of  $t$ , and there is only one stable renormalization group (RG) fixed point. In the limit  $r \rightarrow \infty$ , where  $X(t)$  depends on all the previous variables,  $C(t)$  in model (i) obeys a power law, and the system becomes scale invariant. In model (ii) with  $q \neq 1/2$ , there are two stable RG fixed points, which correspond to the ordered and disordered phases of the information cascade phase transition with critical exponents  $\beta = 1$  and  $\nu_{\parallel} = 2$ .

## 1. Introduction

Collective phenomena have attracted considerable interest and remain an attractive research subject. They are ubiquitous in physical, biological, and social systems.<sup>1-4</sup> Among them, the correlated binary sequence is an important research subject. The correlated random walk, quantum walk and Pólya urn process are examples of the correlated binary sequence.<sup>5-11</sup> The update sequence of spins in the kinetic Ising model is also a correlated binary

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sequence.<sup>12,13</sup> These systems have rich mathematical structures, including scale invariance and phase transitions.<sup>1,14–17</sup>

Scale invariance in a binary sequence is usually defined in terms of the scaling behavior of the variance of displacements with respect to the length.<sup>18,19</sup> If the variance obeys a power law of the length and the exponent is greater than 1, it is called superdiffusive behavior. Superdiffusion can be attributed to long-range positive correlations and can be seen in many strongly correlated binary sequences in nature.<sup>18</sup> Coarse-grained DNA strings, written texts, and financial data are examples. To explain the scale invariance or superdiffusive behavior, a binary stochastic process with a long memory has been proposed.<sup>8,10</sup> It is related to a safety campaign problem. In the Pólya–Friedman urn process,<sup>7,20</sup> the probability for a unit bit in a binary string depends linearly on the fraction  $z$  of the unities preceding it. It shows a dynamical phase transition between normal diffusion phase with an exponent of 1 and the superdiffusion phase with an exponent greater than 1.<sup>8,20</sup> If the memory length is finite and the random variables depend on the recent  $r$  variables,<sup>10</sup> the model is known as Kirman’s ant model.<sup>21,22</sup> It was introduced to describe the switching process of herding behavior in financial markets. In the model, the number of variables taking 1 among the  $r$  variables oscillates randomly.<sup>21,23</sup> On the basis of the model, the key stylized facts in the statistical properties of financial markets have been explained.<sup>24</sup>

The Ising model is a representative example of the phase transition of binary variables.<sup>1</sup> It shows an order-disorder phase transition in the thermodynamic limit. The kinetic Ising model is a single spin update stochastic process of the Ising model.<sup>12</sup> As the system approaches the critical point, the relaxation time diverges, and the dynamic exponent  $Z$  is introduced to classify the universality classes of dynamical critical phenomena in addition to the basic exponents.<sup>1,17</sup> The voting model is a generalized Pólya urn process<sup>25,26</sup> and describes the sequential voting process of subjects (voters) in information cascade experiments<sup>27,28</sup> and a race-track betting market.<sup>29</sup> In an information cascade experiment, there are two options and subjects choose options one by one. They can observe the previous subjects’ choices, and the observations affect their own choices.<sup>30–34</sup> Because there are two options to choose between, the choice sequence becomes a correlated binary sequence.

How the subjects’ choices are affected by the previous subjects’ choices depends on the experimental situation. In the canonical setting of an information cascade experiment, there are two urns,  $R$  and  $B$ , which contain red and blue balls in different proportions.<sup>30–32</sup> Urn  $X \in \{R, B\}$  is fixed at the beginning of each experiment and subjects are asked to choose  $R$  or  $B$ . If the choice coincides with  $X$ , the subject receives a return. Each subject draws a ball from

$X$  and obtains some information because the color of the ball is usually positively correlated with the type of  $X$ . In addition, he also observes the choices of other subjects' before him. If the majority of others' choices suggest  $B$ , he might choose  $B$ , even if his ball suggests that  $R$  is correct. The same situation continues for the subjects after him, and the majority choice of  $B$  tends to continue. This is an information cascade, and it is also called rational herding.

In an experiment with a two-choice quiz, subjects who did not know the answer to a question showed a strong tendency to choose the majority answer.<sup>33</sup> We call a subject who tends to choose the majority choice a herder. On the other hand, the subject who knows the answer chooses the correct choice independently from others' choices, and we call him an independent. How the herder chooses depends on the return structure. If the return is constant, it is rational to choose the majority choice. We call a herder who always chooses the majority choice a digital herder. If the return is inversely proportional to the proportion of subjects who have chosen it, as in the parimutuel payoff odds, it is optimal to choose an option with the same probability as the proportion.<sup>34</sup> We define the optimal herder an analog herder.

The problem to address in studies of information cascades is the convergence of the ratio of correct choices  $z$  in the long sequence limit.<sup>32,35</sup> In the voting model where voters are a mixture of independents and analog herders, the model shows a normal-to-superdiffusion phase transition.<sup>36</sup> If the voters are a mixture of independents and digital herders, an Ising-like phase transition occurs, and the limit value of the variance of  $z$  is the order parameter.<sup>37</sup> If the ratio of herders is low,  $z$  converges to a value, and the order parameter takes 0. We call the phase a one-peak phase because the distribution of  $z$  has only one peak. If the ratio is high,  $z$  converges to one of two values. The value to which  $z$  converges is determined by a probabilistic process, and the order parameter takes a finite and positive value. We call the phase a two-peak phase and the transition between these two phases an information cascade phase transition. In the experiment with a two-choice quiz, the ratio of herders is controlled by the difficulty of the question. We have reported that the phase transition should occur by extrapolating the variance of  $z$ .<sup>33</sup>

In this paper, we study the critical properties of correlated binary stochastic processes based on finite-size scaling (FSS). In particular, our interest lies in the universality classes of information cascade phase transitions. In equilibrium statistical mechanics, the correlation length plays a key role because it is the unique measure of the collective behavior.<sup>1</sup> In the study of the critical behavior of correlated binary sequences, the temporal correlation length  $\xi$  should play the same role as in other nonequilibrium phase transitions.<sup>16,17</sup> For the definition of  $\xi$ , we adopt the second moment correlation time of the autocorrelation function.<sup>38</sup> We

study the scaling properties of  $\xi$  and the integrated correlation time  $\tau$ . We characterize the phases and critical properties of the system by their limit behaviors and critical exponents. We derive some scaling relations and estimate the critical exponents, which clarify the universality classes of the information cascade phase transitions. In addition, we provide a new data analysis method for information cascade experiments. We obtain the finite-size correcting expression for the order parameter. Using it, we hope to solve the convergence problem without referring to any specific model.

The paper is organized as follows. In Section 2, we define the  $r$ -th Markov binary process  $X(t) \in \{0, 1\}, t \in \{1, 2, \dots\}$ , which is a mixture of an independent random variable that takes 1 with probability  $q$  and a random variable that depends on the ratio  $z$  of the variables taking 1 among recent  $r$  variables. We consider two models, analog and digital. We introduce the FSS ansatz for the stochastic process. We adopt the second moment correlation time of the autocorrelation function  $C(t) = \text{Cov}(X(1), X(t+1))$  as the temporal correlation length  $\xi$ . In Section 3, we analyze the stochastic process for finite  $r$ .  $C(t)$  decays exponentially, and there is only one stable renormalization group (RG) fixed point at  $\lim_{t \rightarrow \infty} \xi(t)/t = 0$ . We study the  $r \rightarrow \infty$  limits of the two models in Section 4.  $C(t)$  in the analog model obeys a power law, and  $\lim_{t \rightarrow \infty} \xi(t)/t$  is finite. Regarding the digital model for  $q \neq 1/2$ , there are two stable RG fixed points corresponding to the two phases of the information cascade phase transition. We obtain the scaling relations and estimate the critical exponents. Section 5 presents a summary and future problems. In the Appendices, we derive some results in the main text and show the numerical procedure.

## 2. Models and finite-size scaling analysis

### 2.1 Models

We define the  $r$ -th Markov binary processes  $X(t) \in \{0, 1\}, t \in \{1, 2, \dots, T\}$ .  $X(t+1)$  is a mixture of an independent random variable that takes 1 with probability  $q$  and a random variable that depends on the ratio  $z(t, r)$  of the previous  $r$  variables  $X(s), s \in \{t-r+1, t-r+1, \dots, t\}$ , which takes 1.

$$z(t, r) = \begin{cases} \frac{1}{r} \sum_{s=t-r+1}^t X(s) & \text{for } t \geq r, \\ \frac{1}{t} \sum_{s=1}^t X(s) & \text{for } t < r. \end{cases} \quad (1)$$

If  $t < r$  or in the limit  $r \rightarrow \infty$ ,  $X(t+1)$  depends on all the previous  $t$  variables. It is a simple model for the sequential voting process in information cascade experiments, and the two types of random variable correspond to the two types of voters: independents and herders. If voter

$t$  is an independent, the probability for the correct choice or the independent variable takes 1 is  $q$ . If voter  $t$  is a herder, he obtains information from the previous  $r$  subjects by referring to  $z(t-1, r)$ . If  $t = 1$ , there is no available information, and the choice is random. If  $t > 1$  and under the condition  $z(t-1, r) = z$ , the probability that herder  $t$  chooses the correct option or the dependent variable takes 1 is given by the function  $f(z)$ . The ratio of the independent and dependent variables is  $1-p : p$ . The probability that  $X(t)$  takes 1 is then given by

$$\Pr(X(t+1) = 1 | z(t, r) = z) = (1-p) \cdot q + p \cdot f(z). \quad (2)$$

The first term comes from the independent random variable with the ratio  $1-p$ , and the second term comes from the dependent random variable with the ratio  $p$ .  $\Pr(X(t+1) = 0 | z(t, r) = z)$  is given as  $1 - \Pr(X(t+1) = 1 | z(t, r) = z)$ . For the function  $f(z)$ , we consider two types: (i) digital  $[\theta(z - 1/2)]$  and (ii) analog  $[z]$ . Here  $\theta(z)$  is a Heaviside function with the convention  $\theta(0) = 1/2$ .

## 2.2 Finite-size scaling ansatz

We discuss the scaling property of stochastic binary processes  $\{X(t)\}$ . The critical behavior is governed by a temporal length scale, which we call the correlation length  $\xi$ . The definition of  $\xi$ , we adopt the second moment correlation time of the autocorrelation function  $C(t)$ . according to the definition of the second moment correlation length for spin models.<sup>38</sup>  $C(t)$  is defined as the covariance of  $X(1)$  and  $X(t+1)$ :

$$C(t) \equiv \text{Cov}(X(1), X(t+1)) \equiv E(X(1)X(t+1)) - E(X(1))E(X(t+1)). \quad (3)$$

Here, the expectation value  $E(A)$  of some quantity  $A$  is defined as the ensemble average over the paths of the stochastic process. We denote the  $n$ -th moment of  $C(s)$  for the period  $s < t$  as  $M_n(t)$ .

$$M_n(t) \equiv \sum_{s=0}^{t-1} C(s)s^n.$$

Hereafter,  $t$  in  $M_n(t)$  plays the role of the time horizon or system size of the stochastic process. The second moment correlation time, or correlation length,  $\xi(t)$  is defined as

$$\xi(t) \equiv \sqrt{M_2(t)/M_0(t)}. \quad (4)$$

In the study of equilibrium phase transition of spin models,  $\xi$  is defined as the logarithm of the ratio of the two lowest eigenvalues by the transfer matrix method. When it is difficult to obtain it, the second moment correlation length is adopted as the proxy..<sup>38-41</sup>

The integrated correlation time, or relaxation time,  $\tau(t)$  is defined as

$$\tau(t) \equiv M_0(t)/C(0). \quad (5)$$

$\tau$  and  $\xi$  have the same dimension and one might think that  $\tau$  can play the same role with  $\xi$ . However, the length scale in the critical behaviors of  $\xi$  and  $\tau$  is  $\xi$ , not  $\tau$  and the next scaling relation holds for  $\xi$ .

In FSS, the critical property of any long-time observable  $A(t)$  for the time horizon  $t$  is assumed to be scaled by  $\xi(t)/t$ . We introduce a scale factor  $\sigma$  and the ansatz is written as

$$A(\sigma t)/A(t) = f_A(\xi(t)/t), \quad (6)$$

which is correct up to terms of order  $\xi^{-\omega}$  and  $t^{-\omega}$ .<sup>38</sup> Here  $f_A$  is a universal function and  $\omega$  is a correction-to-scaling exponent. Using the universal function  $f_\xi$  for  $\xi$ , we can extrapolate  $\xi(t)$  for system size  $t$  to the value for system size  $s^k t$  as

$$\xi(\sigma^k t) = \xi(t) \cdot \prod_{l=0}^{k-1} f_\xi(\xi(\sigma^l t)/\sigma^l t). \quad (7)$$

With this information and  $f_A$ ,  $A(t)$  is extrapolated to  $A(\sigma^n t)$  as

$$A(\sigma^n t) = A(t) \cdot \prod_{k=0}^{n-1} f_A(\xi(\sigma^k t)/\sigma^k t). \quad (8)$$

We extrapolate  $A(t)$  for a finite time horizon  $t$  to the limit  $t \rightarrow \infty$  by Eq.(8). We denote the extrapolated values of  $A(t)$  as  $A(\infty)$ .

The statistical errors in  $A(\infty)$  come from three sources: (i) the error in  $A(t)$ , (ii) the error in  $\xi(t)$ , and (iii) the error in  $f_A$  and  $f_\xi$ .<sup>38</sup> When a Monte Carlo method is used to obtain  $A(t)$  and  $\xi(t)$ , these errors are estimated as standard errors. From them, the errors in  $A(\infty)$  are estimated. In this paper, we integrate the master equation of the system and obtain exact estimates of  $A(t)$  and  $\xi(t)$ . There is no statistical error, and only the correction-to-scaling remains in Eq.(6), which will propagate to  $A(\infty)$ . We estimate the error in  $A(\infty)$  as the discrepancy between the extrapolated values from different time horizons  $t$ . If we have the exact value for  $A(\infty)$ , we check the discrepancy between the extrapolated value and the exact value. Please refer to Appendix E for the numerical procedure.

We make a comment about the choice of  $\sigma$ . It is not crucial if one has the exact universal functions. We can even take the limit  $\sigma \rightarrow 1$  and obtain a differential equation for  $A(t)$ . If one adopt a Monte Carlo method to estimate universal functions by Eq.(6),  $\sigma$  should be large enough. Usually and in this paper we take  $\sigma = 2$ .<sup>38</sup>

### 2.3 Three typical cases

In the following sections, we derive  $C(t)$  for the stochastic processes [Eq.(2)] and study their scaling properties. In the models,  $C(t)$  is estimated using

$$C(t) \equiv \text{Cov}(X(1), X(t+1)) = p \cdot \text{Cov}(X(1), f(z(t, r))). \quad (9)$$

We show that there are three asymptotic behaviors of  $C(t)$  for the models. Here, we discuss the scaling properties of  $\xi$  and  $\tau$  for the three cases in advance.

#### (1) Exponential decay case

We assume the following functional form for  $C(t)/C(0)$  with some positive constants  $\delta, p \leq 1$ :

$$C(t)/C(0) = \delta \cdot p^t = \delta \cdot e^{t \log p}. \quad (10)$$

$C(t)$  decays exponentially for  $p < 1$ , and the system is in the disordered phase. The expressions for  $\tau(t)$  and  $\xi(t)$  are

$$\begin{aligned} \tau(t) &= \delta \cdot \frac{1 - p^t}{1 - p}, \\ \xi(t) &= \sqrt{p \frac{(1 - p^t - t^2 p^{t-1}(1 - p))(1 - p) + 2(p + (t - 1)p^{t+1} - tp^t)}{(1 - p)^2(1 - p^t)}}. \end{aligned}$$

In the limit  $t \rightarrow \infty$ ,  $\tau(t)$  and  $\xi(t)$  are finite for  $p < 1$ .

$$\lim_{t \rightarrow \infty} \tau(t) = \frac{\delta}{1 - p} \quad \text{and} \quad \lim_{t \rightarrow \infty} \xi(t) = \frac{\sqrt{p(1 + p)}}{1 - p}. \quad (11)$$

$\xi(t)$  diverges only at  $p = 1$ . The critical exponent  $\nu_{\parallel}$  for  $\xi \propto |p - p_c|^{-\nu_{\parallel}}$  is  $\nu_{\parallel} = 1$ , and  $p_c = 1$ .

The scaling relation might hold in the critical region ( $\lim_{t \rightarrow \infty} \xi(t)/t > 0$ ), and we parametrize  $p$  as  $p = 1 - \frac{s}{t}$ . With  $s$  fixed, we take the limit  $t \rightarrow \infty$ , and  $\tau(t)/t$  becomes

$$\lim_{t \rightarrow \infty} \tau(t)/t|_{p=1-s/t} = \frac{\delta}{s}(1 - e^{-s}) \rightarrow \delta \quad \text{as } s \rightarrow 0. \quad (12)$$

In the same way, we have the limit of  $\xi(t)/t$  in terms of  $s$  as

$$\lim_{t \rightarrow \infty} \xi(t)/t|_{p=1-s/t} = \frac{1}{s} \sqrt{\frac{2 - e^{-s}(2 + 2s + s^2)}{(1 - e^{-s})}} \rightarrow \frac{1}{\sqrt{3}} \quad \text{as } s \rightarrow 0. \quad (13)$$

We obtain the parametric expressions of the universal functions for  $\xi$  and  $\tau$  in terms of  $s$  as follows.

$$f_{\xi}^D(s) \equiv \lim_{t \rightarrow \infty} \frac{\xi(2t)|_{p=1-2s/2t}}{\xi(t)|_{p=1-s/t}} = \sqrt{\frac{(1 - e^{-s})}{(1 - e^{-2s})}} \cdot \frac{2 - e^{-2s}(2 + 4s + 4s^2)}{2 - e^{-s}(2 + 2s + s^2)} \rightarrow 2 \quad \text{as } s \rightarrow 0,$$

$$f_{\tau}^D(s) \equiv \lim_{t \rightarrow \infty} \frac{\tau(2t)|_{p=1-2s/2t}}{\tau(t)|_{p=1-s/t}} = 1 + e^{-s} \rightarrow 2, \text{ as } s \rightarrow 0. \quad (14)$$

Here, the superscript  $D$  indicates “disordered”.

We call the scale transformation  $t \rightarrow 2t$  the RG transformation. Using the extrapolation formulas Eqs.(7) and (8), because  $f_{\xi}^D < 2$  and  $f_{\tau}^D < 2$  for  $s > 0$ , both  $\xi(t)/t$  and  $\tau(t)/t$  vanish in the limit  $t \rightarrow \infty$ , as the following relations hold in the RG transformation.

$$\xi(2t)/2t < \xi(t)/t \quad \text{and} \quad \tau(2t)/2t < \tau(t)/t.$$

The RG stable fixed point is characterized by  $\lim_{t \rightarrow \infty} \xi(t)/t = 0$  and  $\lim_{t \rightarrow \infty} \tau(t)/t = 0$ . It is stable, as the infinitesimally perturbed state does return to the fixed point under the RG transformation.

At  $p = 1$  ( $s = 0$ ), both  $f_{\xi}$  and  $f_{\tau}$  take 2.  $\xi/t$  and  $\tau/t$  are invariant under the RG transformation.

$$\xi(2t)/2t = \xi(t)/t \quad \text{and} \quad \tau(2t)/2t = \tau(t)/t.$$

The fixed point is characterized by  $\lim_{t \rightarrow \infty} \xi(t)/t = 1/\sqrt{3}$  and  $\lim_{t \rightarrow \infty} \tau(t)/t = \delta$ .  $C(t)$  does not decay, and it corresponds to the two-peak phase. Further,  $\delta$  acts as the order parameter of the information cascade transition. Because  $f_{\xi} < 2$  for  $\xi/t < 1/\sqrt{3}$ , the infinitesimal perturbation to the fixed point breaks it. It is an unstable RG fixed point.

## (2) Power-law decay case

We assume that  $C(t)$  behaves asymptotically with some positive constant  $\delta$ ,  $p \leq 1$  as

$$C(t)/C(0) = \delta \cdot t^{p-1}.$$

It is easy to estimate  $\tau$  and  $\xi/t$ ; the results are:

$$\tau(t) \simeq \frac{\delta}{p} \cdot t^p \quad \text{and} \quad \xi(t)/t \simeq \sqrt{\frac{p}{p+2}}. \quad (15)$$

Because  $\xi(t)$  is proportional to  $t$ , all states  $\xi(t)/t \in [0, 1/\sqrt{3}]$  are scale invariant because  $\xi(2t)/2t = \xi(t)/t$  holds.

The universal functions in terms of  $\xi/t$  are

$$\begin{aligned} f_{\xi}^{SI}(\xi/t) &= 2, \\ \log_2 f_{\tau}^{SI}(\xi/t) &= p = \frac{2}{(\xi/t)^2 - 1}. \end{aligned} \quad (16)$$

Here, the superscript  $SI$  indicates “scale invariant.” Because  $f_{\tau}^{SI} < 2$  for  $p < 1$ ,  $\lim_{t \rightarrow \infty} \tau(t)/t = 0$ . At  $p = 1$ ,  $\lim_{t \rightarrow \infty} \tau(t)/t = \delta > 0$  and it is the two-peak phase.

## (3) Two-peak phase case



In the two-peak phase of the information cascade phase transition,  $\lim_{t \rightarrow \infty} C(t) > 0$ . We assume the following asymptotic form for  $C(t)$  with some positive constant  $c$  and a rapidly decaying function  $d(t)$  as

$$C(t)/C(0) = c + d(t). \quad (17)$$

Here,  $c$  is defined as

$$c \equiv \lim_{t \rightarrow \infty} C(t)/C(0).$$

We define  $D_n(t)$  as  $D_n(t) = \sum_{s=0}^{t-1} s^n d(s)$ .  $\tau(t)$  and  $\xi(t)$  are

$$\tau(t) = c \cdot t + D_0(t) \quad \text{and} \quad \xi(t) = \sqrt{\frac{\frac{c}{6}t(t-1)(2t-1) + D_2(t)}{ct + D_0(t)}}. \quad (18)$$

If we assume  $\lim_{t \rightarrow \infty} |D_n(t)/t^{n+1}| \propto 1/t$ , we have

$$\begin{aligned} \tau(t)/t &= c + \frac{D_0(t)}{t} \rightarrow c \quad \text{as } t \rightarrow \infty, \\ \xi(t)/t &= \frac{1}{\sqrt{3}} \sqrt{\frac{ct - \frac{3c}{2} + 3D_2(t)/t^2}{ct + D_0(t)}} \rightarrow \frac{1}{\sqrt{3}} \quad \text{as } t \rightarrow \infty. \end{aligned} \quad (19)$$

These asymptotic behaviors are useful in the estimation of  $c$  and  $\lim_{t \rightarrow \infty} \xi/t$  from the empirical data of correlated binary sequences. The universal functions in the two-peak phase  $f_\xi^{TP} \equiv \xi(2t)/\xi(t)$  and  $f_\tau^{TP} \equiv \tau(2t)/\tau(t)$  are 2 at  $\xi(t)/t = 1/\sqrt{3}$ , as in the previous two cases at  $p = 1$ .  $\xi(t)/t = 1/\sqrt{3}$  is invariant under the RG transformation. The stability of the RG fixed point depends on the shape of the universal function  $f_\xi^{TP}(\xi/t)$  for  $\xi/t \lesssim 1/\sqrt{3}$ .  $f_\xi^{TP}(\xi/t)$  is estimated in the region as

$$f_\xi^{TP}(\xi/t) = 2 + \sqrt{3} \left( \frac{1}{3} - \frac{\xi}{t} \right). \quad (20)$$

As  $f_\xi(\xi/t) > 2$  for  $\xi/t \lesssim 1/\sqrt{3}$ , the fixed point is RG stable.

We make three comments. The first is about the relation between the order parameter of the information cascade phase transition and  $c$ . In our previous work, we adopted the limit value of the variance of the ratio  $z(t) = \sum_{s=1}^t X(s)/t$ , which we denote  $V(z(t))$ .<sup>37</sup> If  $C(s, s') = C$  for  $s, s' \gg 1$ , we have  $\lim_{t \rightarrow \infty} V(z(t)) = \lim_{t \rightarrow \infty} \sum_{1 \leq s, s' \leq t} C(s, s')/t^2 = C$ . Because  $\lim_{t \rightarrow \infty} C(t) = c \cdot C(0)$ , the following relation holds.

$$\lim_{t \rightarrow \infty} V(z(t)) = c \cdot C(0). \quad (21)$$

The second is about another representation for  $c = \lim_{t \rightarrow \infty} C(t)/C(0)$ . We rewrite  $C(t)/C(0)$  as the difference between the conditional probabilities.

$$C(t)/C(0) = \Pr(X(t+1) = 1 | X(1) = 1) - \Pr(X(t+1) = 1 | X(1) = 0)$$

$$= \Pr(X(t+1) = 0|X(1) = 0) - \Pr(X(t+1) = 0|X(1) = 1). \quad (22)$$

If  $c = 0$ , the infinitely departed variable  $\lim_{t \rightarrow \infty} X(t)$  from  $X(1)$  does not depend on  $X(1)$ .  $c$  represents the strength of the information transmission from  $X(1)$  to  $\lim_{t \rightarrow \infty} X(t)$ .

The third is about the definition of the second moment correlation length in eq.(4). In percolation theory, correlation function is defined as the probability that the two sites separated by distance  $t$  are in the same cluster.<sup>42</sup> In the context, the second moment correlation length represents the typical size of the cluster. For spin models, the second moment correlation length for spin-spin correlation function represents the typical size of spin cluster.<sup>39</sup> As  $C(t)$  is defined as the correlation between  $X(1)$  and  $X(t+1)$  in eq.(3), we can regard  $\xi(t)$  as the typical size of spin cluster originated from the first voter.

### 3. Models for $r < \infty$

#### 3.1 Exact results for $r = 1$

For  $r = 1$ , the analog and digital models are the same, and they are known as a correlated random walk.<sup>6</sup> The probability that  $X(t+1)$  takes 1 depends on  $z(t, 1) = X(t)$  as

$$\Pr(X(t+1) = 1|X(t) = x) = (1-p) \cdot q + p \cdot x. \quad (23)$$

We summarize the results for several quantities that are necessary for discussing the scaling relations. The derivations are given in Appendix A. The expectation value of  $X(t)$  is

$$E(X(t)) = q + p^t \left( \frac{1}{2} - q \right), \quad (24)$$

and the variance of  $X(t)$  is  $V(X(t)) = E(X(t)) \cdot (1 - E(X(t)))$ .  $X(t)$  exponentially converges to  $q$  as  $t$  increases.

We put  $f(z(t, 1)) = X(t)$  in Eq.(9) and derive the recursive relation for  $C(t)$ .

$$C(t) = p \cdot \text{Cov}(X(1), X(t)) = p \cdot C(t-1).$$

$C(t) = C(0) \cdot p^t$ , and the system is in the disordered phase for  $p < 1$ . In the limit  $p \rightarrow 1$ , all the variables are completely correlated, and the system is in the two-peak phase with  $c = 1$ .

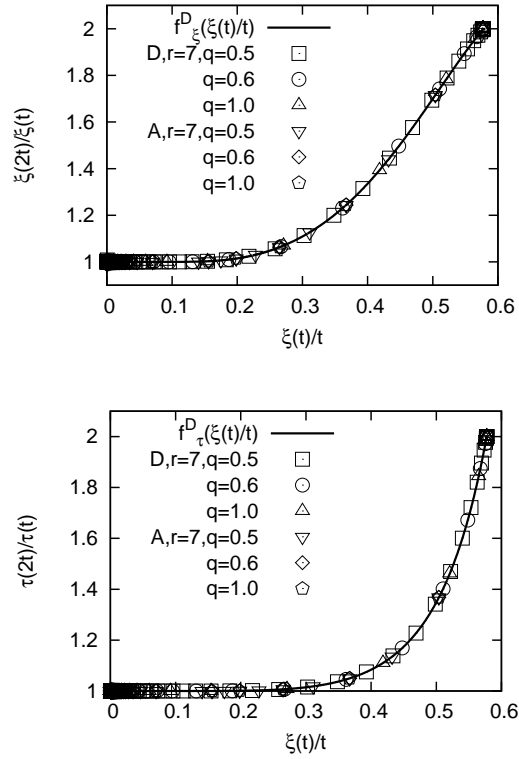
The convergence of  $V(z(t))$  to zero depends on  $q$  for  $p < 1$ . The convergence exponent  $\gamma_z$  is defined as  $V(z(t)) \propto t^{-\gamma_z}$ . The system is in the one-peak phase if  $\gamma_z > 0$  and in the two-peak phase if  $\gamma_z = 0$ . For  $q \neq 1$ ,  $V(z(t))$  obeys the usual power law  $t^{-1}$  and  $\gamma_z = 1$ . For  $q = 1$ ,  $V(z(t)) \propto t^{-2}$  and  $\gamma_z = 2$ . The system is in the one-peak phase and  $c = 0$  for  $p < 1$ .  $c$  changes discontinuously from  $c = 0$  to  $c = 1$  at  $p = 1$ . The critical exponent  $\beta$  for  $c \propto |p - p_c|^\beta$  is zero.

We comment on the relation between  $\gamma_z$  and the dynamic exponent  $Z$ . Space and time are different in nature, and  $Z$  is the scaling exponent between them.<sup>16</sup> The coordinate  $S(t)$  for

the binary process  $\{X(s) \in \{0, 1\}, s \in \{1, \dots, t\}$  is defined as  $S(t) = \sum_{s=1}^t (2X(s) - 1)$ . The variance of  $S(t)$  is given as  $V(S(t)) = 4t^2 \cdot V(z(t)) \propto t^{2-\gamma_z} \propto t^{2/Z}$ . The relation between  $\gamma_z$  and  $Z$  is  $Z = \frac{2}{2-\gamma_z}$ . For normal diffusion,  $V(S(t)) \propto t$ , and  $Z = 2$ .

### 3.2 FSS analysis for finite $r$

We study the scaling properties of the models for  $r = 7$  and  $q \in \{0.5, 0.6, 1.0\}$ . We numerically integrate the models and estimate  $C(t)$  as in Appendix E. We calculate the ratios  $\xi(2t)/\xi(t)$  and  $\tau(2t)/\tau(t)$  for the time horizon  $t = 4 \times 10^3$ . We plot them versus  $\xi(t)/t$  in Fig. 1.



**Fig. 1.**  $\xi(2t)/\xi(t)$  and  $\tau(2t)/\tau(t)$  versus  $\xi(t)/t$ . Symbols indicate different models. We set  $t = 4 \times 10^3$  and  $r = 7$ . Curves are universal functions  $f_\xi^D$  and  $f_\tau^D$  for the discorced Case, which are given in Eq. (14).

All the data for the two models with  $r = 7$  are on the curve of the universal functions  $f_\xi^D(\xi/t)$ ,  $f_\tau^D(\xi/t)$  and can be used to estimate  $\xi(\infty)$  and  $\tau(\infty)$ . The critical behaviors of the models with finite  $r$  are the same as in the exponential decay case. Below, we examine the difference between the analog and digital models with finite  $r$ . In particular, we focus on the critical behavior of  $\xi(\infty)$  for  $p \lesssim p_c = 1$ .

### 3.3 Analog model with finite $r$

We put  $f(z(t, r)) = z(t, r)$  in Eq.(9). For  $t \leq r$ ,  $z(t, r) = z(t) = \frac{t-1}{t}z(t-1) + \frac{1}{t}X(t)$ . Because  $X(t) \propto p \cdot z(t-1)$ , we have the recursive relation.

$$C(t) = \frac{t-1+p}{t} \cdot C(t-1) \quad \text{for } t \leq r.$$

We solve the recursive relation and obtain

$$C(t)/C(0) = \prod_{s=1}^t \frac{s-1+p}{s} \propto t^{p-1}. \quad (25)$$

For  $t \geq r+1$ ,  $z(t, r) = z(t-1, r) + \frac{1}{r}(X(t) - X(t-r))$ . Because  $X(t-r) \propto p \cdot z(t-r-1, r)$ , we have the next recursive relation.

$$C(t) - C(t-1) = \frac{p}{r} \cdot (C(t-1) - C(t-r-1)) \quad \text{for } t \geq r+1.$$

On the basis of the results of FSS analysis, we assume exponential decay for  $C(t) \propto p_r^t$ . We see that  $p_r$  obeys the following relation.

$$p_r^r(p_r - 1) = \frac{p}{r}(p_r^r - 1). \quad (26)$$

In the critical region where  $p = 1 - \epsilon$  for some small positive number  $\epsilon > 0$ ,  $p_r$  can be written as  $p_r = 1 - \frac{\epsilon}{r}$ , and  $\lim_{r \rightarrow \infty} p_r = 1$ . Because  $\xi(\infty) \propto 1/(1-p) = 1/\epsilon$  for the exponential decay case,  $\xi(\infty)$  in the analog model with finite  $r$  behaves as  $\xi(\infty) \propto \frac{r}{\epsilon}$ .  $\xi(\infty)$  diverges linearly with  $r$ .  $\nu_{\parallel}$  is 1 and does not depend on  $r$ .

### 3.4 Digital model with finite $r$

We map the digital model  $(r, q, p)$  to the model with  $(r = 1, q_r, p_r)$ .<sup>43</sup> We recall the probabilistic rule of the model.

$$\Pr(X(t) = 1 | z(t-1, r) = z) = (1-p) \cdot q + p \cdot \theta(z - 1/2).$$

To facilitate the treatment, we modify the model so that  $X(s)$ ,  $s \in \{r(t-1) + 1, \dots, rt\}$  obeys the following rule for  $t \geq 1$ .

$$\Pr(X(s) = 1 | z(r(t-1), r) = z) = (1-p) \cdot q + p \cdot \theta(z - 1/2).$$

All the  $r$  variables  $X(s)$ ,  $s \in \{r(t-1) + 1, \dots, rt\}$  are affected by the same  $z(r(t-1), r)$ . We group the  $r$  variables in a new variable  $X_r(t)$  through  $z(rt, r)$  by the relation

$$X_r(t) = \theta(z(rt, r) - 1/2) = \theta\left(\frac{1}{r} \sum_{s=1}^r X(r(t-1) + s) - 1/2\right). \quad (27)$$

This is a real space renormalization transformation<sup>19</sup> and  $X_r(t)$  depends only on the previous  $X_r(t-1)$ . We write the probabilistic rule for  $X_r(t)$  as

$$\Pr(X_r(t) = 1 | X_r(t-1) = x) = (1 - p_r) \cdot q_r + p_r \cdot x.$$

If  $X_r(t-1) = 0$ , the probability that  $X_r(t)$  takes 1 is  $(1 - p_r)q_r$ . In the modified dynamics of  $X(s)$ , the probability is given as

$$(1 - p_r)q_r = \pi_r((1 - p)q). \quad (28)$$

Here,  $\pi_r(x)$  is defined as  $\pi_r(x) = \sum_{n=(r+1)/2}^r rC_n \cdot x^n (1-x)^{r-n}$ . Likewise, we estimate  $(1 - p_r)(1 - q_r)$  as  $\pi_r((1 - p)(1 - q))$ , and we have the following explicit expressions for  $q_r, p_r$ :

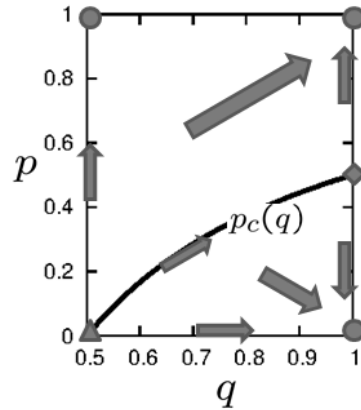
$$q_r = \frac{\pi_r((1 - p)q)}{\pi_r((1 - p)q) + \pi_r((1 - p)(1 - q))}, \quad (29)$$

$$p_r = 1 - (\pi_r((1 - p)q) + \pi_r((1 - p)(1 - q))). \quad (30)$$

For large  $r$ ,  $\pi_r(x)$  behaves as

$$\lim_{r \rightarrow \infty} \pi_r(x) \simeq \begin{cases} 1 & x > 1/2, \\ 1/2 & x = 1/2, \\ 0 & x < 1/2. \end{cases} \quad (31)$$

We study the transformation  $(q, p) \rightarrow (q_r, p_r)$ , which has five fixed points.



**Fig. 2.** Plot of transformation  $(q, p) \rightarrow (q_r, p_r)$ . Arrows indicates the direction of movement from  $(q, p)$  to  $(q_r, p_r)$ . There are three stable fixed points at  $(1, 1)$ ,  $(1, 0)$ ,  $(1/2, 1)$  (filled circles) and two unstable fixed points at  $(1/2, 0)$ ,  $(1, 1/2)$  (filled triangle and diamond, respectively).

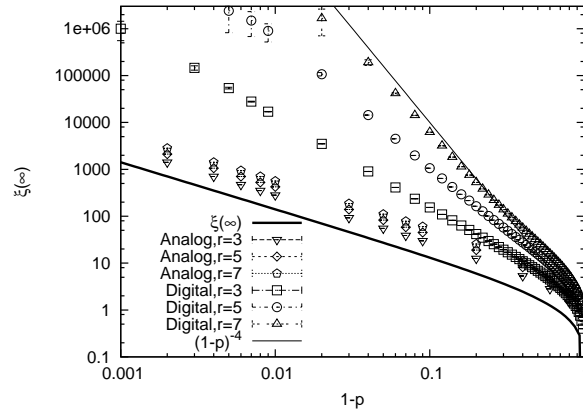
When  $q = 1/2$ ,  $\pi_r((1 - p)q) = \pi_r((1 - p)(1 - q))$  and  $q_r = 1/2$ . For  $p > 0$ ,  $\lim_{r \rightarrow \infty} \pi_r((1 - p)q) = \lim_{r \rightarrow \infty} \pi_r((1 - p)(1 - q)) = 0$  because  $(1 - p)q < 1/2$  and  $(1 - p)(1 - q) < 1/2$ . We have

$\lim_{r \rightarrow \infty} p_r = 1$ . If  $p = 0$ ,  $\pi_r(q) = \pi_r(1 - q) = 1/2$ , and we have  $p_r = 0$ . The fixed points of the transformation are  $(1/2, 0)$  and  $(1/2, 1)$ . The former fixed point is unstable, and the latter is stable on the  $p$  axis.

When  $q > 1/2$ ,  $(q, p)$  on the line  $p(q) = 1 - 1/2q$  moves along it under the transformation because  $(q_r, p_r)$  also satisfies  $p_r = 1 - 1/2q_r$ . We call this line the critical line and denote  $p_c(q) = 1 - 1/2q$ . Because  $(1 - p)q = 1/2$  and  $(1 - p)(1 - q) < 1/2$  on the critical line, we have  $\lim_{r \rightarrow \infty} (q_r, p_r) = (1, 1/2)$ . If  $p < p_c(q)$ , because  $(1 - p)q > 1/2$  and  $(1 - p)(1 - q) < 1/2$ , we have  $\lim_{r \rightarrow \infty} (q_r, p_r) = (1, 0)$ .  $(1, 0)$  is a stable fixed point. In the region  $p > p_c(q)$ ,  $(1, 1)$  is also a stable fixed point. To study the critical region  $p \lesssim 1$ , we write  $p = 1 - \epsilon$ .  $\pi_r(\epsilon q) \propto (\epsilon q)^{(r+1)/2}$ , and  $\pi_r(\epsilon(1 - q)) \propto (\epsilon(1 - q))^{(r+1)/2}$ . With these expressions, we obtain  $q_r \simeq q^{(r+1)/2} / (q^{(r+1)/2} + (1 - q)^{(r+1)/2})$  and  $1 - p_r \propto \epsilon^{(r+1)/2}$ . In the limit  $r \rightarrow \infty$ ,  $(q_r, p_r)$  converges to  $(1, 1)$ , which suggests that  $(1, 1)$  is stable under the transformation.  $\nu_{\parallel}$  is estimated as  $(r + 1)/2$ .

We summarize the results in Fig.2. Under the transformation,  $(1, 1)$ ,  $(1, 0)$  and  $(1/2, 1)$  are stable, and  $(1/2, 0)$  is unstable.  $(1, 1/2)$  has one stable and one unstable direction. The critical properties of the digital model with finite  $r$  are almost the same as those of the disordered case except for  $\nu_{\parallel} = (r + 1)/2$ .

### 3.5 Critical behavior of $\xi(\infty)$ versus $r$



**Fig. 3.**  $\xi(\infty)$  versus  $1 - p$ .  $\xi(\infty)$  is the extrapolated value of  $\xi(t)$  for the analog and digital models with  $t = 4 \times 10^3$ ,  $q = 0.6$ , and  $r \in \{3, 5, 7\}$ . For  $r = 1$ , we plot Eq.(11) with a thick solid line. The thin solid line represents  $(1 - p)^{-4}$  for comparison with the digital model with  $r = 7$ . The error bars are the absolute values of the difference in the extrapolated values from different time horizons,  $4 \times 10^3$  and  $2 \times 10^3$ .

We compare the dependence of the critical behavior of  $\xi(\infty)$  on  $r$  in the two models. We show a double logarithmic plot of  $\xi(\infty)$  versus  $1 - p$  for  $r \in \{3, 5, 7\}$  in Fig. 3. When  $\xi(\infty)$

approaches  $10^6$ , the error bars become large, and one sees that  $t = 4 \times 10^3$  is not sufficient to exceed  $\xi = 10^6$ . For  $r = 1$ ,  $\xi(\infty)$  is given by Eq.(11), and it diverges as  $\propto (1 - p)^{-1}$ . In the analog model with  $r, \nu_{\parallel} = 1$ . In the digital model with  $r, \nu_{\parallel} = (1 + r)/2$ . For  $r = 7$ ,  $\nu_{\parallel} = 4$ , which is consistent with the absolute value of the slope in the figure.

#### 4. $r \rightarrow \infty$ limits of models

In the previous section, we showed that  $p_r$  approaches 1 with  $r$  in the critical region  $p \lesssim 1$  and that  $\xi(\infty)$  diverges in the limit  $r \rightarrow \infty$  for both the analog and digital models. However, the nature of the divergence differs greatly. In the analog model,  $\xi(\infty)$  diverges linearly with  $r$ . In the digital model,  $\xi(\infty) \propto (1 - p)^{-(r+1)/2} \propto e^{r(-\log(1-p)/2)}$ , and  $\xi(\infty)$  diverges exponentially with  $r$ . In this section, we show that the difference leads to completely different critical behaviors in the limit  $r \rightarrow \infty$ .

##### 4.1 Analog model in the limit $r \rightarrow \infty$

We recall some results for the analog model in the limit  $r \rightarrow \infty$ .<sup>8,20,36</sup> The model shows the normal-to-superdiffusion phase transition, if  $q \neq 1$ . For  $p > p_s = 1/2$ ,  $\gamma_z = 2 - 2p < 1$ , and the system is in the superdiffusion phase. If  $p < p_s = 1/2$ ,  $\gamma_z = 1$ , and it is in the normal diffusion phase; further, at  $p = p_s$ , there is a logarithmic correction to the scaling of  $V(z(t))$ . If  $q = 1$ ,  $\gamma_z = 2 - 2p$ , and  $\gamma$  changes continuously with  $p$ ; the phase transition does not occur. The two-peak phase of the information cascade phase transition exists at  $p = 1$ , and for  $p < 1$ , it is in the one-peak phase.

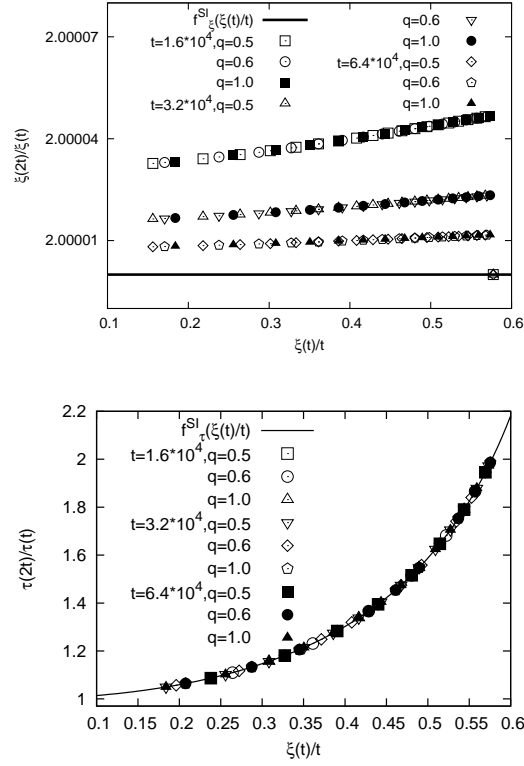
The results are summarized as

$$\lim_{t \rightarrow \infty} V(z(t)) \propto \begin{cases} t^{-1} & p < p_s = 1/2, q \neq 1, \\ t^{-1} \log t & p = p_s, q \neq 1, \\ t^{2p-2} & p > p_s, q \neq 1, \\ t^{2p-2} & q = 1 \\ 1/4 & p = 1 \end{cases} \quad (32)$$

When we take the limit  $r \rightarrow \infty$  in Eq.(25),  $C(t)$  shows a power law decay  $C(t) \sim t^{p-1}$  for large  $t$ . The system is scale invariant, and the scaling relations for  $\xi$  and  $\tau$  are given in Eq. (16).

##### 4.2 FSS analysis of the analog model in the limit $r \rightarrow \infty$

We calculate the ratios of  $\xi$  and  $\tau$  for the time horizon  $t \in \{1.6 \times 10^4, 3.2 \times 10^4, 6.4 \times 10^4\}$  and  $q \in \{0.5, 0.6, 1.0\}$ . We plot the ratios as functions of  $\xi(t)/t$  in Fig. 4. The top panel shows



**Fig. 4.**  $\xi(2t)/\xi(t)$  and  $\tau(2t)/\tau(t)$  versus  $\xi(t)/t$  for the analog model in the limit  $r \rightarrow \infty$ . Symbols and lines indicate different time horizons  $t$  or  $q$  or universal functions  $f_{\xi}^{SI}, f_{\tau}^{SI}$  in Eq.(16). We adopt  $t \in \{1.6 \times 10^4, 3.2 \times 10^4, 6.4 \times 10^4\}$ ,  $q \in \{0.5, 0.5, 1.0\}$

$\xi(2t)/\xi(t)$  versus  $\xi(t)/t$ . The ratio is almost two for any  $\xi(t)/t$ , and only a small correction-to-scaling to  $f_{\xi}^{SI} = 2$  appears up to terms of order  $1/t$ . The bottom panel shows  $\tau(2t)/\tau(t)$  versus  $\xi(t)/t$ , and the ratios are on the curve of  $f_{\tau}^{SI}$  in Eq.(16).

We comment on the relation between  $C(t) = C(1, t+1) \propto t^{p-1}$  and the behavior of  $V(z(t))$  in Eq.(32). If  $C(t, t')$  is assumed to depend on  $t, t'$  through the difference  $|t - t'|$  as  $C(t, t') \propto |t - t'|^{p-1}$ ,  $V(z(t))$  behaves as  $t^{p-1}$ .

$$V(z(t)) = \sum_{1 \leq s, s' \leq t} C(s, s')/t^2 \sim \int_0^t (t-s) \cdot s^{p-1} ds/t^2 \sim t^{p-1}.$$

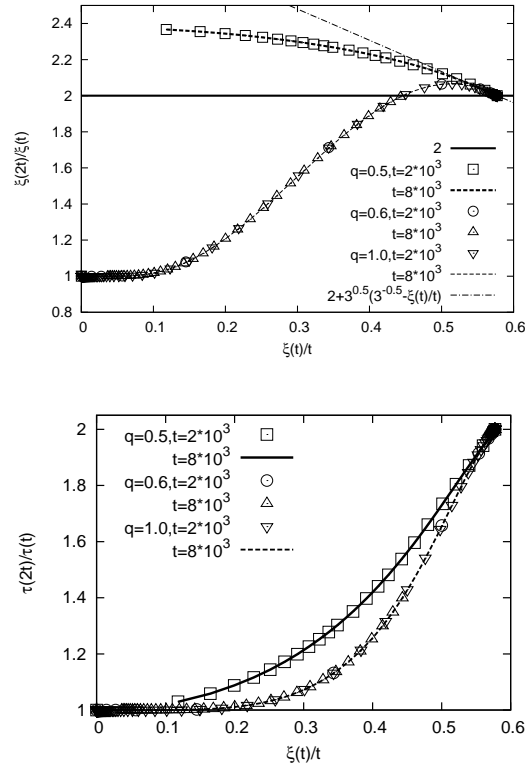
We have  $\gamma_z = p - 1$ . The results in Eq.(32) suggest that the assumption of the translational invariance is wrong.  $C(t, t')$  has a more complex structure, and we study the relation between  $C(t, t')$  and  $V(z(t))$  in Appendix C.



### 4.3 Digital model in the limit $r \rightarrow \infty$

We recall some exact results for the digital model in the limit  $r \rightarrow \infty$ .<sup>37</sup> The model shows the phase transition between the one-peak and two-peak phases as  $p$  passes the critical value  $p_c(q) = 1 - 1/2q$ . Note that  $p_c(q)$  is the same as the critical line for the digital model with finite  $r$ . In the one-peak phase, the probability distribution of  $z(t)$  has a peak at  $z = z_+ \equiv (1 - p)q + p$  in the limit  $t \rightarrow \infty$ . In the two-peak phase, there are two peaks, at  $z = z_- \equiv (1 - p)q$  and  $z_+$ . The probability that  $z(t)$  converges to the lower peak at  $z_-$  is a continuous function of  $p$  and takes a positive value for  $p > p_c$ . The derivative of the probability at  $p_c$  becomes discontinuous in the limit  $t \rightarrow \infty$ , which indicates that the phase transition for  $q > 1/2$  is a continuous nonequilibrium phase transition.

### 4.4 FSS analysis of the digital model in the limit $r \rightarrow \infty$



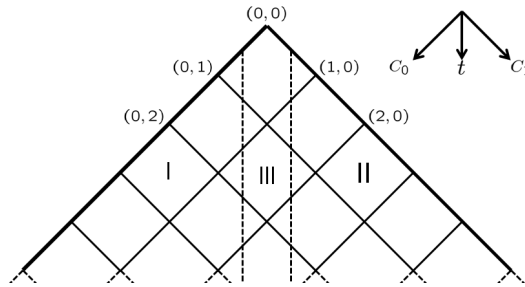
**Fig. 5.**  $\xi(2t)/\xi(t)$  and  $\tau(2t)/\tau(t)$  versus  $\xi(t)/t$  for the digital model in the limit  $r \rightarrow \infty$ . Symbols and lines indicate different time horizons  $t$  or  $q$ . We adopt  $t \in \{2 \times 10^3, 8 \times 10^3\}$  and  $q \in \{0.5, 0.6, 1.0\}$ . We also plot Eq.(20) in the upper figure.

We calculate the ratios of  $\xi$  and  $\tau$  for the time horizons  $t \in \{2 \times 10^3, 8 \times 10^3\}$  and  $q \in \{0.5, 0.6, 1.0\}$  and plot the results in Fig.5. The scaling relation for  $\xi(t)$  indicates that there are

two stable RG fixed points at  $\xi/t = 0$  and  $\xi/t = 1/\sqrt{3}$  for  $q > 1/2$ . Further,  $p_c$  is determined by  $\xi(2t)/\xi(t) = 2$ , and the system is in the two-peak (one-peak) phase for  $p > p_c$  ( $p < p_c$ ). We determine  $p_c$  with the numerical data for  $t = 8 \times 10^3$  and recover the exact result  $p_c = 1 - 1/2q$  to an accuracy of 1%. For  $p > p_c$ ,  $\lim_{t \rightarrow \infty} \xi(t)/t = 1/\sqrt{3}$ . For  $p < p_c$ , because  $\xi(2t)/\xi(t) < 2$ ,  $\lim_{t \rightarrow \infty} \xi(t)/t = 0$ , and  $\lim_{t \rightarrow \infty} \xi(t)$  is finite. At  $p = p_c$ ,  $\xi(2t)/\xi(t) = 2$  and the system is scale invariant. We find  $\tau(2t)/\tau(t) = 2^{1/2}$  and  $\xi(t)/t = 1/\sqrt{5}$  at  $\xi(2t)/\xi(t) = 2$ . For  $q = 1/2$ , there is only one stable RG fixed point at  $\xi/t = 1/\sqrt{3}$ . For  $p > 0$ , the system is in the two-peak phase, and the fixed point at  $\xi/t = 0$  is unstable.

From the scaling relation for  $\tau$  in the bottom panel, for  $p > p_c(q)$ ,  $\tau(2t)/\tau(t)$  converges to 2 in the limit  $t \rightarrow \infty$ ;  $\lim_{t \rightarrow \infty} \tau(t)/t$  takes a positive value, and it is  $c$  in Eq.(19). In the two-peak phase,  $\lim_{t \rightarrow \infty} \xi(t)/t = 1/\sqrt{3}$ , and  $\lim_{t \rightarrow \infty} \tau(t)/t = c > 0$ . For  $p < p_c$ ,  $\tau(\infty)$  is finite, and  $\lim_{t \rightarrow \infty} \tau(t)/t = 0$ . If  $q = 1/2$ ,  $\lim_{t \rightarrow \infty} \xi(t)/t = 1/\sqrt{3}$  for  $p > 0$ , and  $\lim_{t \rightarrow \infty} \tau(2t)/\tau(t) = 2$ . We find  $\lim_{t \rightarrow \infty} \tau(t)/t = c > 0$  for  $p > 0$ .

#### 4.5 Scaling analysis of $C(t)/C(0)$



**Fig. 6.** Directed path representation of  $\{X(t)\}$  as  $(C_1, C_0) = (m, n)$ .

We estimate  $C(t)/C(0)$  for  $q = 1$  using our previous result.<sup>37</sup> We introduce a tilted two-dimensional square lattice  $(m, n)$ ,  $m, n \in \{0, 1, \dots\}$  with the origin at the top of it. We describe the stochastic process as a directed path on the lattice (Fig.6). We map  $X(t)$ ,  $t \in \{1, 2, \dots\}$  to a path on the lattice as  $(m, n) = (C_1, C_0)$  with  $C_1 = \sum_{s=1}^t X(s)$  and  $C_0 = t - C_1$ . The path starts from the top  $[(0, 0)]$ , and if  $X(t) = 1(0)$ , it moves to the lower right (left). We refer to the regions  $m < n$ ,  $m > n$ , and  $m = n$  as I, II, and III, respectively. The probabilistic rules are

summarized as

$$\Pr(X(t) = 0) = \begin{cases} (1-p)(1-q) + p \equiv A & m < n, I, \\ (1-p)(1-q) + p/2 \equiv B & m = n, III, \\ (1-p)(1-q) \equiv C & m > n, II. \end{cases} \quad (33)$$

Now we set  $q = 1$ , and the wall  $n = m - 1$  in II becomes an absorbing wall for the path. If a path enters II, it cannot return to III. To estimate  $C(t)/C(0)$  in Eq.(22), one needs to know  $\Pr(X(t+1) = 0|X(1) = 0)$  as  $\Pr(X(t+1) = 0|X(1) = 1) = 0$  for  $q = 1$ . We denote the number of paths from  $(0, 0)$  to  $(s, s)$  in I and III that enter III  $k$  times as  $A_s^k = k(2s-k-1)!/s!(s-k)!$ .<sup>37</sup> The probability that a path starts from  $(0, 1)$  and reaches  $(s, s)$  is then given as

$$\Pr(z(t = 2s) = 1/2|X(1) = 0) = \sum_{k=1}^s A_s^k \cdot p^{s-1} (1-p)^s / 2^{k-1}. \quad (34)$$

We denote the probability that a path starts from  $(0, 1)$  and stays in region  $J \in \{I, II, III\}$  at  $t$  as  $P_J(t|X(1) = 0)$ . For large  $t$ , the  $k = 1$  term in Eq.(34) is dominant, we have

$$P_{III}(t = 2s|X(1) = 0) \simeq \frac{2}{p} \cdot \frac{e}{8\sqrt{\pi}} s^{-3/2} (4p(1-p))^s \text{ for } t \gg 1. \quad (35)$$

The probability that a path enters II from  $(s, s)$  to  $(s+1, s)$  is  $(1-p/2)$ , and we have the following expression.

$$P_{II}(t = 2s+1|X(1) = 0) = \sum_{u=1}^s P_{III}(2u|X(1) = 0) \cdot (1-p/2). \quad (36)$$

Because  $P_I(t|X(1) = 0) \simeq 1 - P_{II}(t|X(1) = 0)$  for large  $t$  and  $\Pr(X(t+1) = 0) = p$  for  $z(t) < 1/2$ , we have

$$C(t)/C(0) = p \cdot P_I(t|X(1) = 0) = p \cdot (1 - P_{II}(t|X(1) = 0)) \text{ for } t \gg 1.$$

The limit value  $\lim_{t \rightarrow \infty} C(t)/C(0)$ , which we denote as  $c(q = 1, p)$ , is then estimated using the results in<sup>37</sup> as

$$c(q = 1, p) = \lim_{t \rightarrow \infty} C(t)/C(0) = \begin{cases} 0 & p \leq 1/2, \\ \frac{4p-2}{(1+p)} & p \geq 1/2. \end{cases}$$

Using  $c(q = 1, p)$  we rewrite the expression for  $C(t)/C(0)$ :

$$C(t = 2s)/C(0) = c(1, p) + p \cdot \sum_{u=s+1}^{\infty} P_{III}(2u|X(1) = 0) \cdot (1-p/2). \quad (37)$$

Putting the asymptotic form Eq.(35) for  $P_{III}(2u|X(1) = 0)$  in the expression, we obtain

$$C(t)/C(0) \sim c(1, p) + p \cdot \int_t^{\infty} ds \frac{e}{8\sqrt{\pi}} s^{-3/2} e^{-s/\xi(p)} \cdot (1-p/2).$$

Here, we define  $\xi(p) = -/\log \sqrt{4p(1-p)}$ . Using the results, we estimate the critical exponents. We expand  $c(1, p)$  around  $p = p_c(1) = 1/2$ ; then we have  $\beta = 1$  for  $C(1, p) \propto |p - 1/2|^\beta$ . For  $\nu_{\parallel}$ , we expand  $\xi(p)$  around  $p_c = 1/2$  and obtain  $\nu_{\parallel} = 2$ . At  $p = p_c$ ,  $C(t)$  behaves as  $t^{-\alpha}$  with  $\alpha = 1/2$  as  $\xi(p)$  diverges.

The FSS results suggest that the phase transition for  $q > 1/2$  is governed by the fixed point  $(1, 1/2)$  of the transformation  $(q, p) \rightarrow (q_r, p_r)$ . In particular, at  $p = p_c(q)$ ,  $\Pr(X(t) = 0) = 1/2$  in I ( $m < n$ ), and it is a simple symmetric random walk. In II and III,  $\Pr(X(t) = 1) > 1/2$ , and it is not symmetric. The nonrecurrence probability that a simple random walk does not return to the diagonal up to  $t$  behaves as  $t^{-1/2}$ , which can be checked by Eq.(34).  $C(t)$  is proportional to the probability that the random walker remains in I because it is difficult for the random walker to return from II to I even for  $q < 1$ . Thus,  $C(t) \propto P_I(t|X(1) = 0) \sim t^{-1/2}$  holds generally. Furthermore, because the paths of the random walker in I are concentrated around  $z \sim 1/2$ , the same asymptotic behaviors  $\Pr(X(t') = 0|X(t) = 0) \sim |t' - t|^{-1/2}$  should hold for  $t, t' > 0$ . This suggests that the translational invariance is recovered at  $p = p_c$  and  $C(t, t') \propto |t - t'|^{-1/2}$ . From the discussion in 4.2, we have  $\gamma_z = 1/2$ . However, there are many assumptions in the discussion, so  $\gamma_z = 1/2$  should be studied numerically.

The above directed path picture suggests the relation between the information cascade phase transition and a phase transition to absorbing states. For  $q = 1$ , the wall  $n = m - 1$  in II becomes an absorbing wall, and a path does not return to I if it touches the wall. Region II becomes an absorbing state for the path.  $C(t)$  is the survival probability of the path in I, which is the order parameter of the phase transition into absorbing states.<sup>16</sup> We can assume the following scaling form for  $C(t)$  in the critical region for  $q > 1/2$ .

$$C(t)/C(0) \propto t^{-\alpha} \cdot g(t/\xi(t)), \text{ and } \alpha = 1/2. \quad (38)$$

Here, we assume that  $C(t)$  is scaled by  $\xi(t)$  with a universal scaling function  $g$ . With this expression, we can derive the relation between  $\tau(\infty)$  and  $\xi(\infty)$  as  $\tau(\infty) \propto \xi(\infty)^{1-\alpha}$  for  $p \lesssim p_c(q)$ . For  $p \lesssim p_c(q)$ ,  $\xi(\infty)$  should diverge as  $|p - p_c(q)|^{-\nu_{\parallel}}$ , and we have the relation  $\nu_{\tau} = (1 - \alpha)\nu_{\parallel}$  for  $\nu_{\tau}$  of  $\tau(\infty) \propto |p - p_c(q)|^{-\nu_{\tau}}$ . In the limit  $t \rightarrow \infty$  for  $p \gtrsim p_c(q)$ ,  $\lim_{t \rightarrow \infty} C(t)/C(0) = c > 0$ . Because  $g(x)$  should behave as  $g(x) \sim x^{\alpha}$  to cancel  $t^{-\alpha}$ , we have

$$c \propto \xi^{-\alpha} \propto |p - p_c(q)|^{\alpha\nu_{\parallel}}.$$

We have the scaling relation  $\beta = \nu_{\parallel} \cdot \alpha$ .

We can also estimate  $\nu_{\parallel}$  by the relation  $p_r$  in Eq.(30). Because  $(p, q) = (1/2, 1)$  is a fixed

point under the transformation  $X \rightarrow X_r$ , the following relation holds near the fixed point.

$$\lim_{r \rightarrow \infty} \frac{1}{r} |p - 1/2|^{-\nu_{\parallel}} = \lim_{r \rightarrow \infty} |p_r - 1/2|^{\nu_{\parallel}}.$$

With Eq. (30), we have

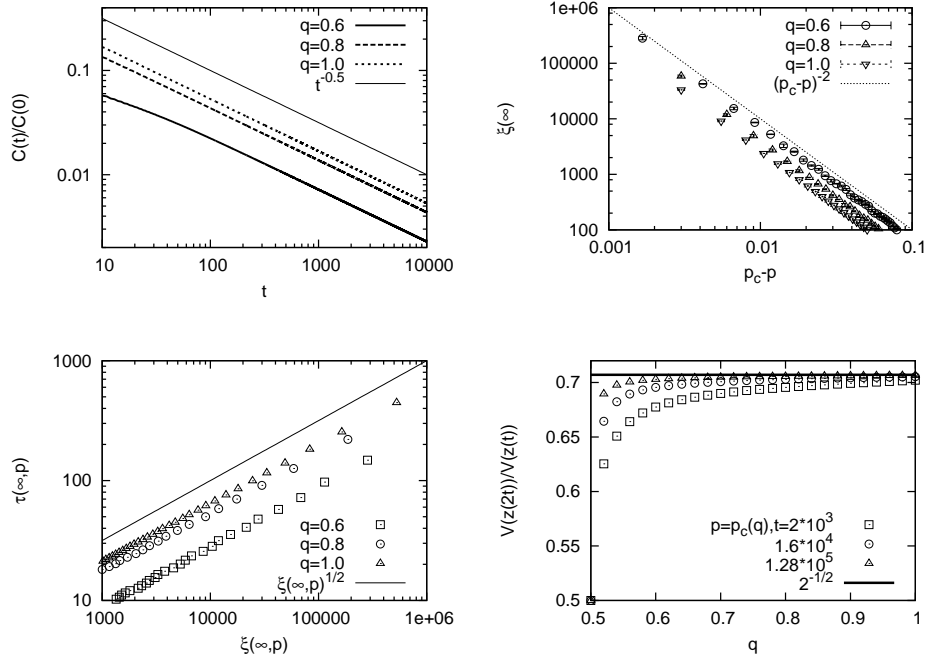
$$(p_r - 1/2) \simeq \sqrt{\frac{2r-2}{\pi e^2}} (p - 1/2).$$

Then we can estimate  $\nu_{\parallel}$  as

$$\nu_{\parallel} = \lim_{r \rightarrow \infty} 1 / \log_r \sqrt{\frac{2r-2}{\pi e^2}} = 2.$$

The result,  $\beta = 1$ ,  $\alpha = 1/2$  and  $\nu_{\parallel} = 2$ , is consistent with the scaling relation  $\beta = \nu_{\parallel} \cdot \alpha$ . These scaling relations suggest that only two exponents,  $\beta$  and  $\nu_{\parallel}$ , might be sufficient to characterize the universality class of the information cascade phase transition.

#### 4.6 Critical behaviors



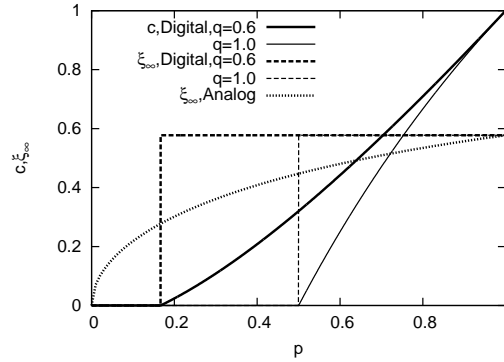
**Fig. 7.** Critical behaviors of  $C(t)$ ,  $\xi(\infty)$ ,  $\tau(\infty)$ , and  $V(z(t))$ . Top left:  $C(t)/C(0)$  versus  $t$  for  $p = p_c(q)$ . Top right:  $\xi(\infty)$  versus  $p_c(q) - p$ . Bottom left:  $\tau(\infty)$  versus  $\xi(\infty)$  in the critical region  $p \simeq p_c(q)$ . We adopt  $q \in \{0.6, 0.8, 1.0\}$ . Bottom right:  $V(z(2t))/V(z(t))$  versus  $q$  at  $p = p_c(q)$ . We adopt  $t \in \{2 \times 10^3, 1.6 \times 10^4, 1.28 \times 10^5\}$ .

We study the critical behaviors of the digital model in the limit  $r \rightarrow \infty$ . Figure 7 shows the results. The top left panel shows  $C(t)/C(0)$  versus  $t$  for  $q \in \{0.6, 0.8, 1.0\}$  and  $p = p_c(q)$ . One sees that  $C(t)$  obeys the power law decay  $t^{-1/2}$ . The top right panel shows  $\xi(\infty)$  versus

$p_c - p$  for  $q \in \{0.6, 0.8, 1.0\}$  and  $p \lesssim p_c(q)$ . The critical exponent  $\nu_{\parallel}$  for  $\xi \propto |p - p_c|^{-\nu_{\parallel}}$  is  $\nu_{\parallel} = 2$ . The lower left panel shows  $\tau(\infty)$  versus  $\xi(\infty)$  for  $p \lesssim p_c$ .  $\nu_{\tau}/\nu_{\parallel}$  is  $1/2$ . The lower right panel shows  $V(z(2t))/V(z(t))$  versus  $q$  for  $p = p_c(q)$ . We have no way to extrapolate the results for finite  $t$  to  $\infty$ . To see the limit  $t \rightarrow \infty$ , we show the results for  $t \in \{2 \times 10^3, 1.6 \times 10^4, 6.4 \times 10^4\}$ . The plot suggests that the ratio is  $1/\sqrt{2}$  in the limit  $t \rightarrow \infty$  for  $q > 1/2$ . Because  $\gamma_z = -\lim_{T \rightarrow \infty} \log_2 V(z(2T))/V(z(T))$ , we have  $\gamma_z = 1/2$ .

## 5. Summaries and Conclusions

In this paper, we performed a FSS analysis of the  $r$ -th Markov binary processes and studied the critical behaviors. As the temporal correlation length  $\xi$ , we propose to use the second moment correlation time of the autocorrelation function  $C(t)$ . As the stochastic process, we consider a mixture of an independent random variable and a dependent random variable that depends on the ratio  $z$  of recent  $r$  variables taking 1. The behavior of the latter dependent variable is defined by  $f(z)$ . We consider two types of  $f(z)$ : (i) analog  $f(z) = z$  and (ii) digital  $f(z) = \theta(z - 1/2)$ .



**Fig. 8.**  $c$  and  $\xi_{\infty} \equiv \lim_{t \rightarrow \infty} \xi(t)/t$  versus  $p$ .

We obtained the following results, some of which are summarized in Fig. 8 and Table I. We denote the limit value  $\lim_{t \rightarrow \infty} \xi(t)/t$  as  $\xi_{\infty}$ . Figure 8 shows  $\xi_{\infty}$  and  $c$  versus  $p$ . Table I summarizes the critical exponents of the information cascade phase transition.

- Temporal correlation length  $\xi$

The critical behavior of the models obeys the FSS relation based on  $\xi$ .  $\xi$  plays an essential role in the phase transitions and critical behaviors of the systems.

- Models with  $r < \infty$

**Table I.** Summary of critical exponents for information cascade phase transition. We denote the analog (digital) model with finite  $r$  as  $A^r(D^r)$  and the limit  $r \rightarrow \infty$  of  $A^r(D^r)$  as  $A^\infty(D^\infty)$ .

Model	$p_c$	$\beta$	$\alpha$	$\nu_{\parallel}$	$\nu_{\tau}$	$Z$	$\gamma_z$
$A^r$	1	0	0	1	1	1	0
$D^r$	1	0	0	$(r+1)/2$	$(r+1)/2$	1	0
$A^\infty$	1	0	0	NA	NA	1	0
$D^\infty, q > 1/2$	$1-1/2q$	1	$1/2$	2	1	$4/3$	$1/2$

The scaling properties are described by those of the disordered system where  $C(t)$  decays exponentially.<sup>43</sup> The RG stable fixed point is at  $\xi_\infty = 0$ , and the system is in the one-peak disordered phase ( $c = 0$ ) with normal diffusive behavior for  $q \neq 1$ . At  $p = 1$ , it is in the two-peak phase ( $c = 1$  and  $\xi_\infty = \sqrt{1/3}$ ).

- Analog model in the limit  $r \rightarrow \infty$

The scaling properties are described by those of the scale-invariant system with  $C(t) \propto t^{p-1}$ . The phase of the system is described by  $\xi_\infty = \sqrt{p/(2+p)}$  and  $c = 0$ . At  $p = 1$ , it is in the two-peak ordered phase ( $c = 1$  and  $\xi_\infty = \sqrt{1/3}$ ).

- Digital model in the limit  $r \rightarrow \infty$

There are two stable and one unstable RG fixed points for  $q \neq 1/2$ . The two stable fixed points correspond to the one-peak disordered phase ( $\xi_\infty = 0$  and  $c = 0$ ) and the two-peak phase ( $\xi_\infty = 1/\sqrt{3}$  and  $c > 0$ ). For  $q = 1/2$ , there is one stable RG fixed point at  $\xi_\infty = 1/\sqrt{3}$ , which corresponds to the two-peak phase with  $c > 0$ .  $\xi_\infty = 0$  is an unstable RG fixed point.

- Finite-size correcting expressions for  $\tau/t$  and  $\xi/t$  in Eq.(19)

We comment on future problems. The first problem is the study of a system in the general  $f(z)$  case. In our experiment on the information cascade phase transition with a two-choice quiz, the behavior of  $f(z)$  fell between that of the analog and digital models.<sup>33</sup> If we adopt  $f(z) = (\tanh(\lambda(z - 1/2)) + 1)/2$  with the control parameter  $\lambda$ , the system becomes a type of kinetic Ising model in which new spins are added as subjects choose sequentially.<sup>43</sup> The important problem is how the scaling relations and critical properties depend on  $f(z)$  and the limit  $r \rightarrow \infty$ . For a sufficiently slow increase of  $r$  with  $t$ , the system can be equilibrated among recent  $r$  variables. In the limit  $t \rightarrow \infty$  with slowly increasing  $r$ , the critical properties might correspond to those of the kinetic Ising model. If we take the limit  $r \rightarrow \infty$  as  $r = t$  and  $\lambda \rightarrow \infty$ , the system shows the phase transition for the digital model. Between the two

extremes, there might be some rich structure.

The second problem is the application of the scaling analysis to empirical data. In our previous paper, we proposed to use  $\gamma_z$  to detect the information cascade phase transition.<sup>33</sup> Considering that  $\xi$  is the essential quantity for characterizing the phase transition and critical properties, and  $c = \lim_{t \rightarrow \infty} C(t)/C(0)$  is the order parameter, one should study these quantities. In laboratory experiments, the system size  $t$  is severely limited. The finite size correcting expressions in Eq.(19) provide convenient expressions for estimating  $c$  and  $\xi_\infty$ .

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## Appendix A: Model for $r = 1$

We derive some quantities of interest for the model for  $r = 1$ . We write the probabilistic rule for  $X(t + 1)$  using  $X(t)$  as  $\text{Prob}(X(t + 1) = 1) = (1 - p) \cdot q + p \cdot X(t)$ . Evaluating the expectation value, we have

$$E(X(t + 1)) = (1 - p)q + p \frac{1}{t - 1} \cdot E(X(t)).$$

The recursive relation for  $\Delta E(X(t)) = E(X(t)) - q$  is

$$\Delta E(X(t+1)) = p \cdot \Delta E(X(t)) \quad \text{with} \quad \Delta E(X(1)) = p(1/2 - q).$$

By solving it, we obtain the result in Eq.(24)

To estimate the correlation function  $C(t, t') = \text{Cov}(X(t), X(t'))$ , it is convenient to use the transfer matrix. We write the conditional probability  $\text{Pr}(X(s+1) = x_s | X(s) = x_s)$  as the  $2 \times 2$  matrix  $T(x_{s+1}, x_s)$ .

$$T(x_{s+1}, x_s) = \begin{pmatrix} T(1, 1) & T(1, 0) \\ T(0, 1) & T(0, 0) \end{pmatrix} = \begin{pmatrix} (1-p)q + p & (1-p)q \\ (1-p)(1-q) & (1-p)(1-q) + p \end{pmatrix}$$

Multiplying by  $T(x_{s+1}|x_s)$  for  $s = t, \dots, t' - 1, t' > t$ , we obtain the joint probability function  $\text{Pr}(X(t') = x_{t'}, X(t' - 1) = x_{t'-1} | \dots, X(t) = x_t)$ ,

$$\text{Pr}(X(t') = x_{t'}, X(t' - 1) = x_{t'-1}, \dots | X(t) = x_t) = \prod_{s=t}^{t'-1} T(x_{s+1}, x_s).$$

Taking the summation over the intermediate variables  $x_s, s = t + 1, \dots, t' - 1$ , we obtain the conditional probabilities  $\text{Pr}(X(t') = x_{t'} | X(t) = x_t)$  as

$$\text{Pr}(X(t') = x_{t'} | X(t) = x_t) = T^{|t'-t|}(x_{t'}, x_t) = \begin{pmatrix} (1 - p^{|t'-t|})q + p^{|t'-t|} & (1 - p^{|t'-t|})q \\ (1 - p^{|t'-t|})(1 - q) & (1 - p^{|t'-t|})(1 - q) + p^{|t'-t|} \end{pmatrix}$$

We obtain  $C(t, t') = V(X(t)) \cdot p^{|t'-t|}$  by estimating the following equation.

$$C(t, t') = V(X(t))(T^{|t'-t|}(1, 1) - T^{|t'-t|}(1, 0)).$$

$E(z(t))$  is estimated by taking the average of  $E(X(s))$  for  $s = 1, \dots, t$  and we have

$$E(z(t)) = q + \left(\frac{1}{2} - q\right)p \frac{1 - p^t}{(1 - p)t} \simeq q + O(t^{-1}).$$

$V(z(t))$  is the summation of  $C(s, s'), 1 \leq s, s' \leq t$  divided by  $t^2$ .

$$\begin{aligned} V(z(t)) &\equiv \frac{1}{t^2} \sum_{1 \leq s, s' \leq t} C(s, s') \\ &= \frac{1}{t^2} [t \cdot q(1 - q) + (q - 1/2)^2 (2p \frac{1 - p^t}{1 - p} - p^2 \frac{1 - p^{2t}}{1 - p^2})] \\ &+ \frac{2}{t^2} q(1 - q) [\frac{p}{1 - p} (t - 1) - (\frac{p}{1 - p})^2 (1 - p^{t-1})] \\ &+ \frac{2}{t^2} (q - \frac{1}{2})^2 \frac{p}{1 - p} [(2p + p^{t+1}) \frac{1 - p^{t-1}}{1 - p} - p^2 \frac{1 - p^{2(t-1)}}{1 - p^2} - 2p^t (t - 1)] \\ &\simeq \frac{q(1 - q)}{t} \cdot \frac{1 + p}{1 - p} + O(t^{-2}) \end{aligned}$$

From these results, in the limit  $t \rightarrow \infty$ ,  $z(t)$  behaves as

$$z(t) \sim N\left(q, \frac{q(1-q)}{t} \frac{1+p}{1-p}\right).$$

To see the relation with the result in<sup>6</sup> we change the variable from  $z(t)$  to  $S(t)$  using  $S(t) \equiv \sum_{s=1}^t (2X(s) - 1) = 2t \cdot z(t) - t$ . The asymptotic formula for  $S(t)$  is

$$S(t) \sim N\left((2q-1)t, 4t \cdot q(1-q) \frac{1+p}{1-p}\right). \quad (\text{A}\cdot 1)$$

In<sup>6</sup> the probability for a step in the same direction as the previous step is denoted as  $\alpha$ . In our notation,  $\alpha = (1-p)/2 + p = (1+p)/2$  for the symmetric case  $q = 1/2$ . After  $t$  steps, the probability that the random walker stays at  $k$  was estimated as

$$k \sim N(0, t \cdot \alpha/(1-\alpha)).$$

(To obtain this expression, we set  $r = m = t = 0$  in Theorem 3.1 in<sup>6</sup> and change variables from  $n$  to  $t$ .) We set  $\alpha = (1+p)/2$  and  $q = 1/2$  in Eq.(A.1), and we recover the same result.

## Appendix B: Analog model with finite $r$

We write the probabilistic rule for  $X(t)$  using  $z(t-1, r)$  as  $\Pr(X(t) = 1) = (1-p) \cdot q + p \cdot z(t-1, r)$ . The expectation value of  $X(t)$  obeys the following relation.

$$E(X(t)) = \begin{cases} (1-p)q + \frac{p}{t-1} \sum_{s=1}^{t-1} E(X(s)) & t \leq r+1, \\ (1-p)q + \frac{p}{r} \sum_{s=t-r}^{t-1} E(X(s)) & t \geq r+2. \end{cases}$$

Solving the recursive relation, we obtain  $\Delta E(X(t))$  for  $t \leq r+1$  as

$$\Delta E(X(t)) = \prod_{s=1}^{t-1} \left( \frac{s-1+p}{s} \right) \Delta E(X(1)) \quad , \quad \Delta E(X(1)) = p\left(\frac{1}{2} - q\right). \quad (\text{B}\cdot 1)$$

$E(X(t))$  shows a power law convergence to  $q$  with  $\Delta E(X(t)) \propto t^{p-1}$  if  $r$  is large.

For  $t \geq r+2$ , we obtain the following recursive relation.

$$\Delta E(X(t)) - \Delta E(X(t-1)) = \frac{p}{r} (\Delta E(X(t-1)) - \Delta E(X(t-r-1))).$$

We assume the exponential convergence of  $\Delta E(t) \propto p_r^t$  and see that  $p_r$  obeys the same relation as in Eq.(26).

## Appendix C: Analog model in the limit $r \rightarrow \infty$

When we take the limit  $r \rightarrow \infty$  of the result for the analog model with finite  $r$ , we see that  $E(X(t))$  converges to  $q$  with the power law decay  $\Delta E(X(t)) \propto t^{p-1}$ . Likewise,  $C(t)/C(0)$  behaves as  $C(t)/C(0) \propto t^{p-1}$ .

Next, we estimate  $C(t, t') = \text{Cov}(X(t), X(t'))$  for  $t' > t \geq 2$ . We start from the probabilistic

rules for  $X(t), X(t')$ .  $X(t), X(t')$  are coupled through  $z(t-1), z(t'-1)$  and we have the following relation.

$$C(t, t') = p^2 \cdot \text{Cov}(z(t-1), z(t'-1))$$

We decompose  $z(t'-1)$  as  $z(t'-1) = \frac{t'-2}{t'-1}z(t'-2) + \frac{1}{t'-1}X(t'-1)$ . We obtain

$$C(t, t') = p^2 \cdot \left( \frac{t'-2}{t'-1} \cdot \text{Cov}(z(t-1), z(t'-2)) + \frac{1}{t'-1} \text{Cov}(z(t-1), X(t'-1)) \right).$$

Because  $\text{Prob}(X(t'-1) = 1) = (1-p) \cdot q + p \cdot z(t'-2)$ , the following recursive relation holds.

$$C(t, t') = \frac{t'-2+p}{t'-1} \cdot C(t, t'-1) \quad (\text{C}\cdot 1)$$

For  $t' = t+1$ , because  $\text{Cov}(z(t-1), z(t-1)) = V(z(t-1))$ , we have the following relation.

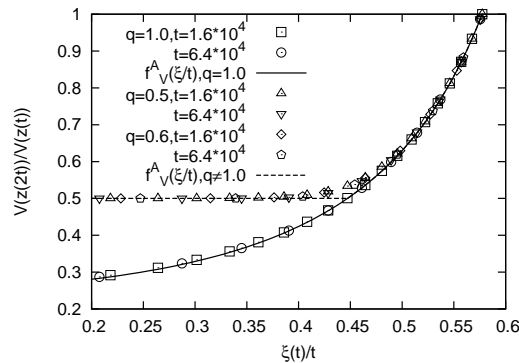
$$C(t, t+1) = p^2 \cdot \frac{t-1+p}{t} \cdot V(z(t-1)).$$

By solving the recursive equation for  $C(t, t')$ , we obtain the following expression.

$$C(t, t') = C(t, t+1) \cdot \prod_{s=t+1}^{t'-1} \frac{s-1+p}{s}.$$

Because  $C(t, t+1) \propto V(z(t))$  and  $\prod_{s=t+1}^{t'-1} \frac{s-1+p}{s} \sim (t'/t)^{p-1}$ , we obtain the asymptotic behavior of  $C(t, t')$  as

$$\lim_{t, t' \rightarrow \infty} C(t, t') \propto V(z(t)) \left( \frac{t'}{t} \right)^{p-1}. \quad (\text{C}\cdot 2)$$



**Fig. C.1.**  $V(z(2t))/V(z(t))$  versus  $\xi(t)/t$  for analog model in the limit  $r \rightarrow \infty$ .

$V(z(t))$  behaves as in Eq. (32) and we can derive the universal function for  $V(z(t))$  as

$$\log_2 f_V^A(\xi(t)/t) = \begin{cases} -1 & \xi(t)/t \leq \frac{1}{\sqrt{5}}, q \neq 1, \\ \frac{4}{(\xi(t)/t)^2 - 1} - 2 & \xi(t)/t > \frac{1}{\sqrt{5}}, q \neq 1, \\ \frac{4}{(\xi(t)/t)^2 - 1} - 2 & q = 1. \end{cases}$$

At  $\xi(t)/t = \frac{1}{\sqrt{5}}$  for  $q \neq 1$ , there is a correction-to-scaling to  $f_V^A$  of order  $\log_2 2$ , which is not negligibly small even for large  $t$ .

We calculate the ratios  $V(z(2t))/V(z(t))$  for the time horizon  $t \in \{1.6 \times 10^4, 6.4 \times 10^3\}$  and  $q \in \{0.5, 0.6, 1.0\}$ . We plot the ratios as functions of  $\xi(t)/t$  in Fig.C.1. Because  $\gamma_z = -\lim_{T \rightarrow \infty} \log_2 V(z(2t))/V(z(t))$  and  $\xi(t)/t$  is scale invariant, the figure shows the relation between  $\gamma_z$  and  $\lim_{t \rightarrow \infty} \xi(t)/t$ . For  $q \neq 1$ , there are two phases: the normal diffusion phase ( $\gamma_z = 1$ ) and the superdiffusion phase ( $\gamma_z < 1$ ). At  $\xi(t)/t = \frac{1}{\sqrt{5}}$ , one sees the correction-to-scaling to  $f_V^A$ , as we have noted previously. If  $q = 1$ ,  $\gamma_z$  changes smoothly from 2 to 0 as  $\xi(t)/t$  increases, and the normal-to-superdiffusion phase transition does not occur.

#### Appendix D: $c(q, p)$ and $\beta$ for digital model in the limit $r \rightarrow \infty$

We have defined  $c$  as the difference between the conditional probabilities in Eq. (22). We denote the probability that a path starts from the wall  $n = m + 1$  in I, crosses the diagonal only once, and reaches the wall  $n = m - 1$  in II as  $R_1$ .  $R_2$  is the probability that the path starts from the wall  $n = m - 1$  in II, crosses the diagonal only once, and reaches the wall  $n = m + 1$  in I. The explicit expressions for  $R_1$  and  $R_2$  are

$$R_1 = \frac{(1 - B)(1 - \sqrt{1 - 4A(1 - A)})}{A(2 - \frac{B}{A}(1 - \sqrt{1 - 4A(1 - A)}))}, \quad (\text{D} \cdot 1)$$

$$R_2 = \frac{B(1 - \sqrt{1 - 4C(1 - C)})}{(1 - C)(2 - \frac{1-B}{1-C}(1 - \sqrt{1 - 4C(1 - C)}))}. \quad (\text{D} \cdot 2)$$

Here,  $A, B, C$  are defined in Eq.(38).<sup>37</sup> With  $R_1$  and  $R_2$ , we can write  $c(q, p)$  as

$$c(q, p) = p \cdot \frac{(1 - R_1)(1 - R_2)}{1 - R_1 R_2}. \quad (\text{D} \cdot 3)$$

For  $p \leq p_c(q)$ ,  $R_1 = 1$ , and we have  $c = 0$ . If  $p > p_c(q)$ ,  $R_1 < 1$ , and  $c$  takes a positive value. On the critical line  $(p_c(q), q)$ ,  $R_1 = 1$ , and  $R_2$  changes smoothly from 1 at  $q = 1/2$  to 0 at  $q = 1$ . At  $p = 1$ , we have  $c = 1$ .

We estimate  $\beta$  for the infinitesimal perturbation  $(p_c(q), q) \rightarrow (p_c(q) + x, q)$  with  $|x| \ll 1$  as  $c \propto x^\beta$ . The expansion of  $R_1$  to first order in  $x$  is

$$R_1 = 1 - \frac{16q^2 x}{4q - 1}.$$

For  $q \neq 1/2$ ,  $R_2 < 1$  and  $p_c(q) > 0$ . In the expansion of  $c = p(1 - R_1)(1 - R_2)/(1 - R_1 R_2)$ ,  $(1 - R_1)$  determines the critical exponents. We obtain  $\beta = 1$ .

### Appendix E: Numerical procedure

We explain the procedure for numerical integration of the master equations. First, we explain the case of  $r \rightarrow \infty$ . We denote the joint probability function for  $\sum_{s=1}^t X(s)$  and  $X(1)$  as  $P(t, n, x_1) \equiv \Pr(\sum_{s=1}^t X(s) = n, X(1) = x_1)$ . For  $t = 1$ ,  $P(1, 1, 1) = (1 - p)q + p/2$  and  $P(1, 0, 0) = (1 - p)(1 - q) + p/2$ . The other components are zero. The master equation for  $P(t, n, x_1)$  is

$$P(t + 1, n, x_1) = q((n - 1)/t) \cdot P(t, n, x_1) + (1 - q(n/t)) \cdot P(t, n, x_1) \quad (\text{E} \cdot 1)$$

Here, we define  $q(z) \equiv (1 - p)q + p \cdot f(z)$ . We impose the boundary conditions  $P(t, n, x_1) = 0$  for  $n < 0$  or  $n > t$ . The unconditional probability function  $P(t, n) \equiv \Pr(\sum_{s=1}^t X(s) = n)$  is estimated as  $P(t, n) = \sum_{x_1} P(t, n, x_1)$ .

The joint probability function  $P(x_{t+1}, x_1) = \Pr(X(t + 1) = x_{t+1}, X_1 = x_1)$  is then estimated as

$$P(x_{t+1}, x_1) = \sum_{n=0}^t P(t, n, x_1) \cdot (q(n/t) \cdot x_{t+1} + (1 - q(n/t)) \cdot (1 - x_{t+1})).$$

Using  $P(x_{t+1}, x_1)$ , we can estimate  $C(t)$  as

$$\begin{aligned} C(t) &= E(X(t + 1)X(1)) - E(X(1)) \cdot E(X(t + 1)) \\ &= P(1, 1) - \sum_{x_{t+1}} P(x_{t+1}, 1) \cdot \sum_{x_1} P(1, x_1). \end{aligned}$$

As for the case  $r < \infty$ , the procedure is slightly complicated. First, we define the joint probability functions,

$$P(t, n, x_t, \dots, x_1) \equiv \Pr(\sum_{s=1}^t X(s) = n, X(t) = x_t, \dots, X(1) = x_1) \text{ for } t \leq r + 1$$

$$P(t, n, x_t, \dots, x_{t-r+1}, x_1) \equiv \Pr(\sum_{s=1}^t X(s) = n, X(t) = x_t, \dots, X(1) = x_1) \text{ for } t \geq r + 2.$$

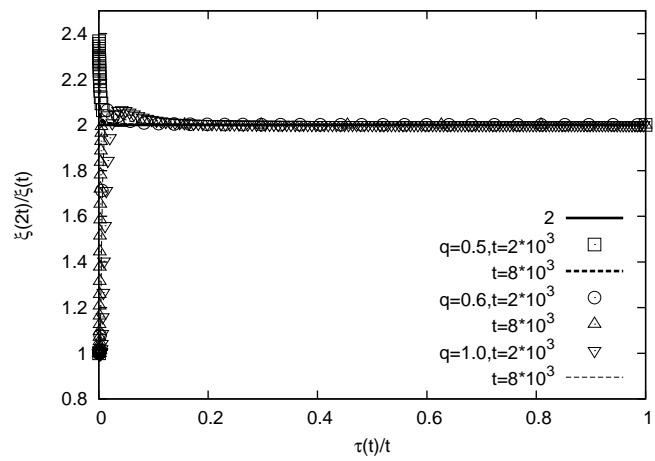
For  $t \leq r + 1$ ,  $P(t, n, x_t, \dots, x_1)$  depends on all  $t$  variables  $\{x_s\}$ ,  $s = 1, 2, \dots, t$ . For  $t \geq r + 2$ ,  $P(t, n, x_t, \dots, x_{t-r+1}, x_1)$  depends on the  $r$  variables  $\{x_s\}$ ,  $s = t - r + 1, 2, \dots, t$  and  $x_1$ . Their master equations are

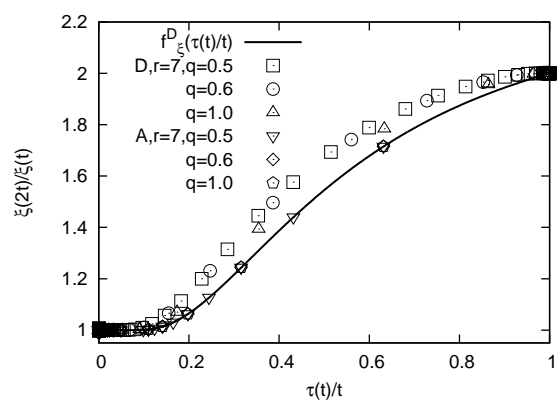
$$\begin{aligned} P(t + 1, n, x_{t+1}, \dots, x_1) &= P(t, n, x_t, \dots, x_1)(1 - q(n/t))(1 - x_{t+1}) \\ &+ P(t, n - 1, x_t, \dots, x_1)q(n/t)x_{t+1} \text{ for } t \leq r. \end{aligned}$$

$$\begin{aligned}
P(t+1, n, x_{t+1}, \dots, x_{t-r+2}, x_1) &= \sum_{x_{t-r+1}} [P(t, n, x_t, \dots, x_{t-r+1}, x_1)(1 - q(n/t))(1 - x_{t+1}) \\
&\quad + P(t, n-1, x_t, \dots, x_{t-r+1}, x_1)q(n/t)x_{t+1}] \text{ for } t \geq r+1.
\end{aligned}$$

We estimate  $P(t, n, x_1)$  from the joint probability functions by summing over the variables except  $n$  and  $x_1$ . We then estimate  $P(x_{t+1}, x_1)$  and  $C(t)$  from  $P(t, n, x_1)$ .







## Brief Notes on Preparing L<sup>A</sup>T<sub>E</sub>X Compuscript for *Journal of the Physical Society of Japan*

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This document briefly provides instructions on how to prepare your manuscript in L<sup>A</sup>T<sub>E</sub>X format. As regards general instructions for preparing manuscripts, please refer to “Instructions for Preparation of Manuscript”, which is available at our Web site (<http://jpsj.jps.jp/>).

### 1. Introduction

You can use this file as a template to prepare your manuscript for *Journal of Physical Society of Japan* (JPSJ).<sup>1,2</sup> No sections or appendices should be given to other categories than Regular Papers. Key words are not necessary.

Copy `jpsj3.cls`, `cite.sty`, and `overcite.sty` onto an arbitrary directory under the `texmf` tree, for example, `$texmf/tex/latex/jpsj`. If you have already obtained `cite.sty` and `overcite.sty`, you do not need to copy them.

Many useful commands for equations are available because `jpsj3.cls` automatically loads the `amsmath` package. Please refer to reference books on L<sup>A</sup>T<sub>E</sub>X for details on the `amsmath` package.

### 2. Options

#### 2.1 Paper type

`jpsj3.cls` has class options for paper types. You should choose the appropriate option listed in Table I. Default (without option) is for a draft.

#### 2.2 Two-column format

The `twocolumn` option may help estimate the length of your manuscript particularly for Letters or Short Notes, which has a limitation of four or two printed pages each. If the

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**Table I.** List of options for paper types.

Option	Paper type
<code>ip</code>	Invited Review Papers
<code>st</code>	Special Topics
<code>letter</code>	Letters
<code>fp</code>	Full Papers
<code>shortnote</code>	Short Notes
<code>comment</code>	Comments
<code>addenda</code>	Addenda
<code>errata</code>	Errata

**Fig. 1.** You can embed figures using the `\includegraphics` command. EPS is the only format that can be embedded. Basically, figures should appear where they are cited in the text. You do not need to separate figures from the main text when you use  $\text{\LaTeX}$  for preparing your manuscript.

`txfonts` package is available in your  $\text{\LaTeX}$  system, you can estimate the length more accurately.

### 2.3 Equation numbers

The `seceq` option resets the equation numbers at the start of each section.

Label figures, tables, and equations appropriately using the `\label` command, and use the `\ref` command to cite them in the text as “as shown in Fig. `\ref{f1}`”. This automatically labels the numbers in numerical order.

### Acknowledgment

For environments for acknowledgement(s) are available: `acknowledgment`, `acknowledgments`, `acknowledgement`, and `acknowledgements`.

### Appendix

Use the `\appendix` command if you need an appendix(es). The `\section` command should follow even though there is no title for the appendix (see above in the source of this file).

For authors of Invited Review Papers, the `profile` command is prepared for the author(s)’ profile. A simple example is shown below.

```
\profile{Taro Butsuri}{was born in Tokyo, Japan in 1965. ...}
```

## References

- 1) The abbreviation for JPSJ must be “J. Phys. Soc. Jpn.” in the reference list.
- 2) More abbreviations of journal titles are listed in “Instructions for Preparation of Manuscript”.